Consistency Conditions for Difference Schemes with Singular Coefficients*

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I. Introduction. In [1], the author considered the stability and convergence of finite-difference approximations to a certain class of partial differential equations containing singular coefficients. It was found, in particular, that straightforward replacement of derivatives by corresponding difference quotients could often lead to difference operators which were unbounded, even with respect to the L_2 norm. This negated any chance of stability, and by the Lax Equivalence Theorem [2], of convergence as well.

One way out of this difficulty, as discussed in [1], is to measure stability and convergence with respect to the sequence of mean pth power norms

$$\|U\|_{M_p} = \left\{\frac{1}{M}\sum_{j=1}^{M} |U_j|^P\right\}^{1/P}, \qquad 1 \leqq p \leqq \infty, \tag{1.1}$$

where $M = M(\Delta t)$ is the number of lattice points of the finite-difference grid.

It is the purpose of the present paper to note that the same sort of troubles can occur when investigating the *consistency* of an approximation to an initial-value problem containing singular coefficients. That is, straightforward replacement of derivatives by difference quotients may often result in schemes which are not consistent in the usual Lax-Richtmyer sense. As an example of this, consider the m-dimensional, spherically symmetric diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{(m-1)}{r}\frac{\partial u}{\partial r}, \qquad 0 \leqq t \leqq T, \tag{1.2}$$

with initial and regularity conditions of the form

$$u(r, 0) = u_0(r) \qquad \text{and} \qquad \partial u(0, t)/\partial r = 0. \tag{1.3}$$

The spatial domain is an m-sphere of radius R, but the precise nature of the boundary conditions there need not concern us. A simple approximation to (1.2) is given by

$$\begin{aligned} v(r, t + \Delta t) &= v(r, t) + \lambda[v(r + \Delta r, t) - 2v(r, t) + v(r - \Delta r, t)] \\ &\quad + ((m - 1)/r)(\lambda \Delta t)^{1/2}[v(r + \Delta r, t) - v(r, t)] \\ &\equiv C(\Delta t)v(r, t), \end{aligned} \tag{1.4}$$

where $\lambda = \Delta t/(\Delta r)^2$. We take $v(r, 0) = u_0(r)$ and utilize the regularity condition to specify that $v(\Delta r - r, t) = v(r - \Delta r, t)$ when $r - \Delta r < 0$.

According to the Lax-Richtmyer theory, if (1.4) is to be consistent with the

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differential equation (1.2) we must have

$$\left\| \left\{ \frac{C(\Delta t) - I}{\Delta t} - \frac{\partial}{\partial t} \right\} u(r, t) \right\| \to 0 \quad \text{as} \quad \Delta t \to 0 \tag{1.5}$$

uniformly in $t \in [0, T]$, for a sufficiently wide class of genuine solutions. Substitution for $C(\Delta t)$ and u_t from (1.4) and (1.2), respectively, will show that

$$\left\{ \frac{C(\Delta t) - I}{\Delta t} - \frac{\partial}{\partial t} \right\} u(r, t) = \frac{(\Delta r)^2}{12} \frac{\partial^4 u}{\partial r^4} + \frac{(m-1)}{2r} \Delta r \frac{\partial^2 u}{\partial r^2} + O[(\Delta r)^4] + O\left[\frac{(\Delta r)^2}{r}\right]$$

if $u(r, t)$ is smooth enough, and utilizing the L_2 norm shows that

$$\left\| \left\{ \frac{C(\Delta t) - I}{\Delta t} - \frac{\partial}{\partial t} \right\} u(r, t) \right\|^2 = \int_0^R \left[\frac{(m-1)}{2r} \Delta r \frac{\partial^2 u}{\partial r^2} \right]^2 dr + \text{remaining terms}\,. \tag{1.6}$$

It is well known (see [3], for example) that the genuine solutions of (1.2) are analytic for $t > 0$ and are such that $\lim\inf u_{rr} > 0$ as $r \to 0$. This means that the function $\Delta r \cdot u_{rr}/r$ appearing in (1.6) will not be in L_2 for any $\Delta r \neq 0$, and it follows that condition (1.5) cannot possibly be fulfilled. We therefore have

LEMMA 1. *The difference approximation* (1.4) *is not consistent with the differential equation* (1.2) *when measured with respect to the* L_2 *norm.*

We do not, however, wish to abandon (1.4) and we shall therefore investigate consistency with respect to the mean pth power norm (1.1).

II. Sufficient Conditions for Mean pth Power Consistency. Let us consider a general initial-value problem of the form $u(0) = u_0$ and

$$u_t = Au = \sum_{\nu=0}^{N} \sum_{k_1+k_2+\cdots+k_Q=\nu} A_{k_1k_2\cdots k_Q} \frac{\partial^\nu u}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_Q^{k_Q}}\,, \tag{2.1}$$

where the $A_{k_1k_2\cdots k_Q}$ are functions of x, and u is a P component vector.

The corresponding finite-difference operator $C(\Delta t)$ must take the form

$$C(\Delta t) = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_Q} c_{k_1k_2\cdots k_Q} T(k_1\Delta x_1, k_2\Delta x_2, \cdots, k_Q\Delta x_Q)\,, \tag{2.2}$$

where the $c_{k_1k_2\cdots k_Q}$ are $P \times P$ matrices and T denotes the translation operator. Since

$$\begin{aligned} T(k_1\Delta x_1, k_2\Delta x_2, \cdots, k_Q\Delta x_Q) &= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left[k_1\Delta x_1 \frac{\partial}{\partial x_1} + k_2\Delta x_2 \frac{\partial}{\partial x_2} + \cdots + k_Q\Delta x_Q \frac{\partial}{\partial x_Q} \right]^\nu \\ &= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left\{ \sum_{l_1+l_2+\cdots+l_Q=\nu} \frac{\nu!(k_1\Delta x_1)^{l_1} (k_2\Delta x_2)^{l_2} \cdots (k_Q\Delta x_Q)^{l_Q}}{(l_1)!(l_2)!\cdots(l_Q)!} \frac{\partial^\nu}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_Q^{l_Q}} \right\} \end{aligned}$$

by the multinomial expansion, we may therefore write

$$C(\Delta t) = \sum_{\nu=0}^{\infty} \sum_{l_1+\cdots+l_Q=\nu} \sum_{k_1} \cdots \sum_{k_Q} \frac{(k_1\Delta x_1)^{l_1} \cdots (k_Q\Delta x_Q)^{l_Q}}{(l_1)! \cdots (l_Q)!} \cdot c_{k_1\cdots k_Q} \frac{\partial^\nu}{\partial x_1^{l_1} \cdots \partial x_Q^{l_Q}}, \tag{2.3}$$

where the interchange in summation is certainly valid for explicit schemes and for all implicit schemes where $c_{k_1k_2\cdots k_Q} \to 0$ fast enough as any of the $k_i \to \pm\infty$.

Substituting (2.1) and (2.3) into the consistency relation (1.5), we obtain for a sufficiently smooth set of genuine solutions

$$\begin{aligned}\left\| \left\{ \frac{C(\Delta t) - I}{(\Delta t)} - A \right\} u(t) \right\|_{M_p} &= \Bigg\| \sum_{\nu=0}^{\infty} \sum_{l_1+\cdots+l_Q=\nu} \Bigg\{ -\frac{I\delta_{\nu 0}}{\Delta t} + \frac{1}{\Delta t} \sum_{k_1} \cdots \sum_{k_Q} \frac{(k_1\Delta x_1)^{l_1} \cdots (k_Q\Delta x_Q)^{l_Q}}{(l_1)! \cdots (l_Q)!} \\ &\qquad \cdot c_{k_1\cdots k_Q} - A_{l_1\cdots l_Q} \Bigg\} \cdot \frac{\partial^\nu u}{\partial x_1^{l_1} \cdots x_Q^{l_Q}} \Bigg\|_{M_p}\end{aligned}$$

where we define $A_{l_1\cdots l_Q} \equiv 0$ if $l_1 + \cdots + l_Q > N$, and where $\delta_{\nu 0}$ is the Kronecker delta ($\delta_{\nu 0} = 1$ if $\nu = 0$, zero otherwise). Choosing some integer $N^* \geqq N$, and using the triangle inequality,

$$\begin{aligned}\left\| \left\{ \frac{C(\Delta t) - I}{\Delta t} - A \right\} u(t) \right\|_{M_p} &\leqq \sum_{\nu=0}^{N^*} \sum_{l_1+\cdots+l_Q=\nu} \\ &\left\| \left\{ -\frac{I\delta_{\nu 0}}{\Delta t} + \frac{1}{\Delta t} \sum_{k_1} \cdots \sum_{k_Q} \frac{(k_1\Delta x_1)^{l_1} \cdots (k_Q\Delta x_Q)^{l_Q}}{(l_1)! \cdots (l_Q)!} c_{k_1\cdots k_Q} - A_{l_1\cdots l_Q} \right\} \right\|_{M} \\ &\cdot \left\| \frac{\partial^\nu u}{\partial x_1^{l_1} \cdots \partial x_Q^{l_Q}} \right\|_{M_p} + \sum_{l_1+\cdots+l_Q=N^*+1} \\ &\left\| \frac{1}{\Delta t} \sum_{k_1} \cdots \sum_{k_Q} \frac{(k_1\Delta x_1)^{l_1} \cdots (k_Q\Delta x_Q)^{l_Q}}{(l_1)! \cdots (l_Q)!} c_{k_1\cdots k_Q} \frac{\partial^{N^*+1} u(\xi)}{\partial x_1^{l_1} \cdots \partial x_Q^{l_Q}} \right\|_{M_p},\end{aligned} \tag{2.4}$$

where ξ is a Q-component vector which lies along the line between the point $(x_1, x_2, \cdots, x_Q)$ and $(x_1 + k_1\Delta x_1, x_2 + k_2\Delta x_2, \cdots, x_Q + k_Q\Delta x_Q)$.

The expression (2.4) permits us to state the

THEOREM. *For a well-posed initial-value problem of the form* (2.1), *suppose for some integer* $N^* \geqq N$ *we have the two conditions*

$$\lim_{\Delta t \to 0} \left\| \left\{ -\frac{I\delta_{\nu 0}}{\Delta t} + \frac{1}{\Delta t} \sum_{k_1} \cdots \sum_{k_Q} \frac{(k_1\Delta x_1)^{l_1} \cdots (k_Q\Delta x_Q)^{l_Q}}{(l_1)! \cdots (l_Q)!} \cdot c_{k_1\cdots k_Q} - A_{l_1\cdots l_Q} \right\} \right\|_{M_p} = 0 \tag{2.5}$$

for all $l_i \geqq 0$ *such that* $l_1 + l_2 + \cdots + l_Q = \nu$ *and* $\nu = 0, 1, 2, \cdots, N^*$, *where* $A_{l_1\cdots l_Q} = 0$ *if* $l_1 + \cdots + l_Q > N$, (2.6) $$\lim_{\Delta t \to 0} \left\{ \frac{(\Delta x_1)^{l_1} \cdot (\Delta x_2)^{l_2} \cdots (\Delta x_Q)^{l_Q}}{\Delta t} \right\} = 0$$

for all $l_1 + l_2 + \cdots + l_Q = N^* + 1$.

Then the finite-difference operator (2.2 *must be consistent with the initial-value problem* (2.1) *when measured with respect to the mean pth power norm.*

III. An Application. As an example consider the difference scheme (1.4) for the diffusion equation. In this case $Q = P = 1$ and we have

(3.1) $$\begin{gathered} A_0 = 0, \quad A_1 = (m-1)/r, \quad A_2 = 1, \\ c_{-1}(j\Delta r) = \lambda, \quad c_0(j\Delta r) = 1 - 2\lambda - (m-1)\lambda/j, \\ c_{+1}(j\Delta r) = (1 + (m-1)/j)\lambda, \end{gathered}$$

where the difference scheme is defined on the lattice $r = j\Delta r$ and $j = j_0, j_0 + 1, \cdots, j_0 + M - 1$ for $0 < j_0 < 1$. Then with (3.1) the sums in (2.5) become,

$$\begin{aligned} \text{for } \nu = 0 &: \frac{-1}{\Delta t} + \frac{1}{\Delta t}\{c_{-1}(j\Delta r) + c_0(j\Delta r) + c_{+1}(j\Delta r)\} = 0, \\ \nu = 1 &: \frac{\Delta r}{\Delta t}\{-c_{-1}(j\Delta r) + c_{+1}(j\Delta r)\} - A_1(j\Delta r) = 0, \\ \nu = N = 2 &: \frac{(\Delta r)^2}{2\Delta t}\{c_{-1}(j\Delta r) + c_{+1}(j\Delta r)\} - A_2(j\Delta r) = \frac{m-1}{2j}. \end{aligned}$$

The second condition (2.6) is fulfilled because $\lim_{\Delta t \to 0} (\Delta r)^3/\Delta t = 0$, and we must therefore investigate the mean pth power norm of the vector $\{(m - 1)/2j\}$, $j = j_0, j_0 + 1, \cdots, j_0 + M - 1$. We cannot take $p = \infty$, for the M_∞ norm of the vector $\{1/j\}$ is simply $1/j_0$, but any finite value of p will do. For example, if $p = 1$, then

$$\left\| \left\{ \frac{1}{j} \right\} \right\|_{M_1} = \left\{ \frac{1}{M} \sum_{j=1}^{M} \left| \frac{1}{j_0 + j - 1} \right| \right\} \sim \frac{\ln M}{M} \to 0 \quad \text{as } \Delta t \to 0,$$

and in general

$$\left\| \left\{ \frac{1}{j} \right\} \right\|_{M_p} = \left\{ \frac{1}{M} \sum_{j=1}^{M} \left| \frac{1}{j_0 + j - 1} \right|^p \right\}^{1/p} = \frac{1}{M^{1/p}} O(1) \to 0$$

$$\text{as } \Delta t \to 0, \qquad 1 < p < \infty.$$

We have therefore established

LEMMA 2. *The difference scheme* (1.4) *is a consistent approximation to the diffusion equation* (1.2) *with respect to any mean pth power norm,* $1 \leqq p < \infty$.

Establishing convergence of (1.4) in the M_p norm is quite another matter, however. If we select midpoint spacing of the finite-difference grid (i.e. $j_0 = \frac{1}{2}$), then the regularity condition (1.3) is enforced by the symmetry condition $v(\Delta r/2, t) = v(-\Delta r/2, t)$, and the corresponding difference matrices will take the form

$$(3.2)\quad C(\Delta t) = \begin{bmatrix} 1-(2m-1)\lambda & (2m-1)\lambda & 0 & 0 & \cdots & 0 \\ \lambda & 1-2\lambda-2(m-1)\lambda/3 & [1+2(m-1)/3]\lambda & 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & \lambda & 1-2\lambda-\frac{2}{2j-1}(m-1)\lambda & \left[1+\frac{2(m-1)}{2j-1}\right]\lambda & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 & \lambda & 1-2\lambda-\frac{2(m-1)}{2M-1}\lambda \end{bmatrix},$$

if it is assumed, for example, that $u(R, t) = 0$. We note that when $0 \leqq \lambda \leqq 1/(2m - 1)$ and $m \geqq 2$ all elements in (3.2) are nonnegative and therefore

$$||C||_{M_\infty} = \max_{(i)} \left\{ \sum_{j=1}^{M} |C_{ij}| \right\} = 1 .$$

This yields

LEMMA 3. *The difference scheme* (1.4) *for* $m \geqq 2$ *with midpoint spacing is stable in the* M_∞ *norm if* $0 \leqq \lambda \leqq 1/(2m - 1)$.

Unfortunately, we do not have the requisite consistency in the M_∞ norm so as to be able to conclude the convergence of (1.4). As (1.2)–(1.3) is well posed, all we can say for now is that if $E(t)$ denotes the solution operator, then

$$(3.3)\qquad ||\{C^n(\Delta t) - E(t)\}u_0||_{M_p} \leqq ||\{C^n(\Delta t) - E(t)\}u_0||_{M_\infty} = O(1)$$

as $\Delta t \to 0$ and $n\,\Delta t \to t$ for any continuous initial function u_0.

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